Integration by Parts

(4.8) Theorem. If u, v are two functions of x, then

$$\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\left(\int v\,dx\right)\right]dx.$$

In words:

Integral of the product of two functions = First function × integral of second function – Integral of (derivative of first function × integral of second function).

Proof. Let $\int v\,dx = w$, then $v = \dfrac{dw}{dx}$

Now, $$\frac{d}{dx}(u\,w) = u\frac{dw}{dx} + w\frac{du}{dx} \tag{1}$$

Integrating both sides of (1) with respect to x, we have

$$\begin{aligned} u\,w &= \int\left(u\frac{dw}{dx} + w\frac{du}{dx}\right)dx \\ &= \int u\frac{dw}{dx}\,dx + \int w\frac{du}{dx}\,dx \\ &= \int u\,v\,dx + \int w\frac{du}{dx}\,dx \end{aligned}$$

i.e., $$\begin{aligned} \int u\,v\,dx &= u\,w - \int\frac{du}{dx}\,w\,dx \\ &= u\int v\,dx - \int\left[\frac{du}{dx}\left(\int v\,dx\right)\right]dx. \end{aligned}$$

Note. The choice of the functions u and v is to be made in such a way that the antiderivative of v is easily found. If antiderivatives of both u and v are known, then v is to be selected as that part which yields $\int\left(\frac{du}{dx}\int v\,dx\right)dx$ easier to be evaluated than $\int\left(\frac{dv}{dx}\int u\,dx\right)dx$. Although there are no general directions for choosing u and v, but if one factor involves power of x, then it is profitable to take this factor as u. Since antiderivates of logarithmic, inverse trigonometric and inverse hyperbolic functions are not obvious, these functions are to be taken as u. If after the choice of u and v, an integral is obtained which is more complicated than the original integral, then our selection of u and v is not appropriate. It is just possible that the rule may not work and we may have to apply some other technique.

Example 7. Evaluate $I = \int x \sin x\,dx$.

Solution. Take $u = x$ and $v = \sin x$ so that $\int v\,dx = \int \sin x\,dx = -\cos x$

and $\dfrac{du}{dx} = 1.$

Now, integrating by parts, we have

$$\begin{aligned} I &= x(-\cos x) - \int 1(-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x. \end{aligned}$$

...uate $\int x^2 (\ln x)^2 \, dx$.

Integration of Rational Functions

(4.10) Definition. An expression of the type $\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials with real coefficient, is called a **rational function.**

In this section we perform the integration of such functions.

Example 15. Evaluate $I = \displaystyle\int \frac{dx}{x^2 + 3x + 4}$

Solution. $I = \displaystyle\int \frac{dx}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}$

Let $x + \dfrac{3}{2} = z$, then $dx = dz$ and on substitution, we have

$$I = \int \frac{dz}{z^2 + \left(\frac{\sqrt{7}}{2}\right)^2} = \frac{2}{\sqrt{7}} \arctan\left[\frac{z}{\sqrt{7}/2}\right]$$

$$= \frac{2}{\sqrt{7}} \arctan\left(\frac{2x+3}{\sqrt{7}}\right).$$

Example 16. Evaluate $I = \displaystyle\int \frac{3x+1}{2x^2 - 2x + 3}\, dx$.

Solution. We split $3x + 1$ into two parts such that one part is derivative of the denominator i.e., $4x - 2$. Then

$$3x + 1 = \frac{3}{4}(4x - 2) + \frac{5}{2} \quad \text{and}$$

$$I = \frac{3}{4}\int \frac{4x-2}{2x^2 - 2x + 3}\, dx + \frac{5}{2}\int \frac{dx}{2\left(x^2 - x + \frac{3}{2}\right)}$$

$$= \frac{3}{4}\int \frac{4x-2}{2x^2 - 2x + 3}\, dx + \frac{5}{4}\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{5}}{2}\right)^2}.$$

$$= \frac{3}{4} \ln|2x^2 - 2x + 3| + \frac{\sqrt{5}}{2} \arctan\left(\frac{2x-1}{\sqrt{5}}\right).$$